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ON THE LENGTH OF BARKER SEQUENCES

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ABSTRACT

A Barker sequence, is a finite binary sequence of integers, each ± 1, whose all non-trivial acyclic autocorrelation coefficients are of size at most 1. It is widely believed that there does not exist any Barker sequence of length greater than 13. in this paper we focus on the Barker sequences with odd length. We fist present a relation for the product of any two consecutive members of such a Barker sequence and then we will show that the length is at most 13

KEYWORDS: Acyclic, Autocorrelation Coefficients, Barker Sequence

1. INTRODUCTION

Given a degree n polynomial $p \in \mathbb{C}[z]$ with complex coefficients, suppose that the set $\{a_0, \cdots, a_n\}$ generates p. For any $k \in \{0, \cdots, n\}$ *the* k *th acyclic autocorrelation coefficient* of p is defined by

$$c_k = \sum_{j=0}^{n-1-k} a_j \bar{a}_{j+k}. \quad (1)$$

For all such values of k we define $c_{-k} = c_k$. it is customary to call the number c_0 the *peak autocorrelation* and the other c_k s the *off-peak autocorrelation* of p. From the definition of c_k, one can easily verify that

$$p(z)p(\frac{1}{z}) = \sum_{k=-n}^{n} c_k z^k,$$

and

$$\|p(z)\|_4 = \{\sum_{k=-n}^{n} c_k^2\}^{\frac{1}{4}}.$$

In many applications it is of the interest when $|a_j| = 1$, in particular when $a_j \in \{-1, +1\}$ for all j and in that case p is respectively called a *unimodular* or *Littlewood* polynomial. In 1953 [1], Barker considered special type of Littlewood polynomials as follows:

DEFINITION

A Littlewood polynomial p *so that all its off-peak autocorrelation have the property* $|c_k| \leq 1$ *is called a Barker polynomial and the set that generates* p *is called a Barker sequence*

If the above property hold for a unimodular polynomials, then it is called *generalized Barker polynomial*. For further information of the subject see $[2-4]$.

Note that we excluded the peak autocorrelation of p in the above definition, because c_0 can never be dominated by any number less than the degree of p. In fact, by (1), we have

$$c_0 = \sum_{j=0}^{n} a_j^2 = n+1.$$

2. BARKER SEQUENCES OF LENGTH n

In the remaining, we only consider the Barker sequences with length n, instead of $n+1$ and so we use the subscript $\{1,\cdots,n\}$ instead of $\{0,\cdots,n\}$. So the picture of (1) is

$$c_k = \sum_{j=1}^{n-k} a_j a_{j+k}. \tag{2}$$

and

$$c_0 = \sum_{j=1}^{n} a_j^2 = n.$$

Up to now only eight different Barker sequences are known. There are two Barker sequences of length 4 and one Barker sequence for each of lengths 2, 3, 5, 7, 11 and 13. In all eight sequences, the first two elements a_1 and a_2 take only the value of $+1$. In what follows, we present these eight Barker sequences.

$n = 2:\{1,1\}$

$3:\{1,1,-1\}$

$4:\{1,1,1,-1\}$

$4:\{1,1,-1,1\}$

$5:\{1,1,1,-1,1\}$

$7:\{1,1,1,-1,-1,1,-1\}$

$11:\{1,1,1,-1,-1,-1,1,-1,-1,1,-1\}$

$13:\{1,1,1,1,1,-1,-1,1,1,-1,1,-1,1\}$

The following table represents the corresponding k th acyclic autocorrelation coefficients for each of the eight Barker sequences.

n	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}
2	1	-	-	-	-	-	-	-	-	-	-	-
3	0	-1	-	-	-	-	-	-	-	-	-	-
4	1 1	0 0	- -	- -	- -	- -	- -	- -	- -	- -	- -	- -
5	0	1	0	1	-	-	-	-	-	-	-	-
7	0	-1	0	-1	0	-1	-	-	-	-	-	-
11	0	-1	0	-1	0	-1	0	-1	0	-1	-	-
12	0	-1	0	1	0	1	0	1	0	1	0	1

Therefore, for each k

$$|c_{n-k}| = \begin{cases} 0, & if \quad k \quad even \\ 1, & if \quad k \quad odd \end{cases} \tag{3}$$

The following lemmas are tools of achieving our main result.

Lemma 1: *(Barker to Barker Transformations) Any of the transformations*

1. $a_i \to (-1)^i a_i$,

2. $a_i \to (-1)^{i+1} a_i$

3. $a_i \to -a_i$

transform a Barker sequence into another Barker sequence.

Lemma 2: *The k th acyclic autocorrelation coefficients of a Barker sequence of length n have the following property*

$$n \equiv c_k + c_{n-k} (mod 4).$$

Proof. Let $\{a_1, \cdots, a_n\}$ be a Barker sequence of length n. Define

$$x = \sum_{i=1}^{n-k} \chi(a_i a_{i+k} = 1)$$

and

$$y = \sum_{i=1}^{n-k} \chi(a_i a_{i+k} = -1),$$

Where χ is the characteristic function. One can easily verify that

$$x + y = n - k, \qquad x - y = c_k,$$

that yield

$y = (n-k-c_k)/2.$

So, letting $d_k := \prod_{i=1}^{n-k} a_i a_{i+k}$, we have

$$d_k = \prod_{i=1}^{n-k} a_i a_{i+k} = 1^x (-1)^y = (-1)^y = (-1)^{(n-k-c_k)/2}$$

and so

$$d_k d_{n-k} = (-1)^{(n-k-c_k)/2} (-1)^{(n-(n-k)-c_{n-k})/2}$$

$$= (-1)^{(n-k-c_k+n-n+k-c_{n-k})/2} \quad (4)$$

$$= (-1)^{(n-c_k-c_{n-k})/2}.$$

On the other hand,

$$d_k d_{n-k} = \prod_{i=1}^{n-k} a_i a_{i+k} \prod_{i=1}^{k} a_i a_{i+n-k} \quad (5)$$

$$= \prod_{i=1}^{n} a_i a_{i+k(modn)} = \prod_{i=1}^{n} a_i^2 = 1.$$

Now, (4) and (5) imply that $(n-c_k-c_{n-k})/2$ is even and hence $n \equiv c_k + c_{n-k} (mod 4).$

Lemma 3: If m is even, then the m th acyclic autocorrelation coefficients of a Barker sequence of odd length n depends only on n and its value is

$$c_m = (-1)^{(n-1)/2}$$

Proof. Let $m = 2j$ and n be odd. Then

i) $|c_k + c_{n-k}| = 1$ for every k,

ii) $c_{2j+1} = 0$, and

iii) $c_{2j} = \pm 1$.

Since by (3) $n \equiv c_k + c_{n-k} (mod 4)$, we have $n \equiv \pm 1 (mod 4)$. If $n \equiv +1 (mod 4)$, then $c_{2j} = +1$ and if $n \equiv -1 (mod 4)$, then $c_{2j} = -1$. So, in any case, we have

$$c_{2j} = (-1)^{(n-1)/2}.$$

Lemma 4: *If* $\{a_1, \cdots, a_n\}$ *is a Barker sequence of odd length* n*, then*

$$a_i a_{i+1} = -(a_{n-i} a_{n-i+1})$$

Proof. By (3) $|c_k + c_{n-k}| = 1$ for every k and so, by the lemma (2), $n \equiv \pm 1(mod\, 4)$. Moreover if d_k is as in the proof of that lemma, then

$$d_k d_{k+1} = (-1)^{(n-k-c_k)/2}(-1)^{(n-(k+1)-c_{k+1})/2}$$

$$= (-1)^{(2n-2k-1-c_k-c_{k+1})/2}$$

$$= (-1)^{n-k-(1+c_k+c_{k+1})/2},$$

and

$$d_k d_{k+1} = (\prod_{i=1}^{n-k} a_i a_{i+k})(\prod_{i=1}^{n-k-1} a_i a_{i+k+1})$$

$$= [(a_1 a_{1+k})\cdots(a_{n-k} a_n)][(a_1 a_{2+k})\cdots(a_{n-k-1} a_n)]$$

$$= (\prod_{i=1}^{n-k-1} a_i^2)(\prod_{i=2+k}^{n} a_i^2)(a_{k+1} a_{n-k})$$

$$= (\prod_{i=1}^{n-k-1} 1)(\prod_{i=2+k}^{n} 1)(a_{k+1} a_{n-k})$$

$$= (a_{k+1} a_{n-k}).$$

The last two calculations yield

$$a_{k+1} a_{n-k} = (-1)^{n-k-(1+c_k+c_{k+1})/2}.$$

Note that if $c_k c_{n-k} = \pm 1$, then $n = \pm 1 + 4m$ for some positive integer m. This because $c_k c_{n-k} \equiv n(mod\, 4)$. Therefore by using lemma (3), we get

$$a_{n-k} a_{k+1} = (-1)^{n-k-(1+c_k+c_{k+1})/2}$$

$$= (-1)^{x+4m-k-(1+x)/2}$$

$$= (-1)^{4m}(-1)^{x-k-(1+x)/2}$$

$$= (-1)^{x-k-(1+x)/2}$$

$$= (-1)^{-k-(1+x-2x)/2}$$

$$= (-1)^{-k+(x-1)/2}$$

$$= (-1)^{2m}(-1)^{-k+(x-1)/2}$$

$$= (-1)^{-k+(x+4m-1)/2}$$

$$= (-1)^{k+(n-1)/2}.$$

Therefore for any two consecutive values of $i \in \{1, \cdots, n\}$, we have $a_i a_{n-i+1} = (-1)^{i-1+(n-1)/2}$ and $a_{i+1}a_{n-i} = (-1)^{i+(n-1)/2}$. Thus

$$a_i a_{i+1} = -(a_{n-i}a_{n-i+1})$$

Lemma 5: *Let* $\{a_1, \cdots, a_n\}$ *be a Barker sequence of odd length* n. *Let* $p \in \{1, \cdots, n\}$ *be so that* $a_{p+1} = -1$ *and* $a_i = 1$, *whenever* $1 \le i \le p$. *Then for any* $p > 1$ *we have*

1. $a_i a_{i+1} = a_{2i}a_{2i+1}$, $1 \le i \le (n-3)/2$
2. $p \le n-2$ implies p is odd
3. If $pj + r \le n-2$ for $1 \le r \le p$, then $a_{p(j-1)+r} = a_{p(j-1)+1}$.

3. THE MAIN RESULT

In the next theorem, we present a new proof for Barker sequences of odd length

Main Theorem: *If* $\{a_1, \cdots, a_n\}$ *is a Barker sequence of odd length* n, *then* $n \le 13$

Proof. By lemma (1), we may assume that $a_1 = a_2 = +1$. In general, we would have peruse the following modes separately.

1. n is grater than any number divisible by 4,
2. n is divisible by 3
3. n is less than any number divisible by 3,
4. n is divisible by 4
5. $3p < n < 4p$, where p is a positive integer.

Since n is an odd number, the fourth case is irrelivant. The first 3 case was proved by Turyn and storer in [5].

Suppose that $3p < n < 4p$. Then $3p \le n-1$ and so by lemma (5), p (and therefore $3p$) is odd. By part 3 of lemma (5), for $j = 2$ and for $1 \le r \le p$, we have

$$p \times 2 + r \le p \times 2 + p \le 3p \le n-2$$

and so

$$a_{(2-1)p+r} = a_{(2-1)p+1}. \quad (6)$$

Hence by assumption, we have $a_i = +1, 1 \le i \le p.$

Since $a_{p+1} = -1$, by (6) we have

$$a_i = -1, p+1 \le i \le 2p.$$

Note that if $a_i = a_{i+1}$, then by lemma (4) $a_{n+1-i} = -a_{n-i}$. Therefore the first two blocks of length p, create the last two blocks of the same length, which are of alternating $+1$'s and -1's. Since the second block and the penultimate block have the common element, we have $n \ge 4p-1$ and as $4p-1 \le n \le 4p$,

$$n = 4p-1. \quad (7)$$

The first and the second blocks of p elements are respectively -1's and $+1$'s. In what follows, we will show that elements of the last two blocks are alternating $+1$'s and -1's. By lemma (4), we have

$$a_p a_{p+1} = -(a_{4p-1-p} a_{4p-1-p+1}) = -(a_{3p-1} a_{3p}).$$

Note that since $a_p a_{p+1} = -1$, the terms a_{3p-1} and a_{3p} have the same signs. Moreover the second block and its penultimate block have a common element which is a_{2p}. Thus

$$-1 = a_{2p} = a_{2p+2} = \cdots = a_{2p+(p-1)}$$

and so

$$a_{2p+1} = a_{2p+3} = \cdots = a_{2p+(p-2)} = +1.$$

With the above descriptions, we can conclude that the sequence is of the form

$$+_1, +_2, \cdots, +_p, -_{p+1}, -_{p+2}, \cdots, -_{2p}, +_{2p+1}, -_{2p+2}, \cdots, -_{3p-1}, -_{3p}, +_{3p+1}, \cdots, -_{4p-1}$$

This means if the sequence is of the form $\{a_i\}_{i=1}^{4p-1}$, then

$$a_i = +1 1 \le i \le p$$

$$a_i = -1 p+1 \le i \le 2p$$

$$a_i = (-1)^{i+1} 2p+1 \le i \le 3p-1$$

$$a_i = (-1)^i 3p \le i \le 4p-1$$

Therefore,

$$c_2 = \sum_{i=1}^{4p-3} a_i a_{i+2}$$

$$= \sum_{i=1}^{p-2} a_i a_{i+2} + a_{p-1}a_{p+1} + a_p a_{p+2} + \sum_{i=p+1}^{2p-2} a_i a_{i+2} + a_{2p-1}a_{2p+1}$$

$$+ \sum_{i=2p}^{3p-3} a_i a_{i+2} + a_{3p-2}a_{3p} + a_{3p-1}a_{3p+1} + \sum_{i=3p}^{4p-3} a_i a_{i+2}$$

$$= (p-2)-1-1+(p-2)-1+(p-2)-1-1+(p-2).$$

So $c_2 = 4(p-2)-5$. But as $c_2 \le 1$, we have $4(p-2)-5 \le 1$ and hence

$$p \le \frac{7}{2}. \quad (8)$$

Among the odd positive numbers, only 3 satisfies (8). Hence $p = 3$ and we get can construct the sequence of length $4 \times 3 - 1 = 11$.

CONCLUSIONS

A glory in the proof of our Main Theorem is applying "Barker into Barker" transformations described in Lemma (1). There are other proofs that the number 13 dominates all barker sequences of odd length and non of them are similar to the proof we presented.

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